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Rellich Type Inequalities with Weights in Plane Domains

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Abstract—We determine some special functionals as sharp constants in integral inequalities for test functions, defined on plane domains. First we prove a new one dimensional integral inequality. Also, we prove some generalizations of a classical Rellich result for two dimensional case, when there is an additional restriction for Fourier coefficients of the test functions. In addition, we examine a Rellich type inequality in plane domains with infinite Euclidean maximal modulus. As an application of our results we present a new simple proof of a remarkable theorem of P. Caldiroli and R. Musina from their paper "Rellich inequalities with weights", published in Calc. Var. **45** (2012), 147–164.

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1. INTRODUCTION

The original inequality of F. Rellich (see [1]) deals with test functions in the domain $\Omega = \mathbb{R}^d \setminus \{0\}$ of the Euclidean space $\mathbb{R}^d$. In the case $d = 2$ the Rellich inequality becomes to be non-trivial for functions with vanishing first Fourier coefficients, only.

There are many generalization of Rellich's result for the Laplace operator and polyharmonic operators in $\Omega = \mathbb{R}^d \setminus \{0\}$ (see [2–8]). Also, there is a few papers on the Rellich type inequalities considered for test functions in domains $\Omega \neq \mathbb{R}^d \setminus \{0\}$ (see [9–12]).

In the book [13] by A.A. Balinsky, W.D. Evans and R.T. Lewis the reader may find the basic results on the Hardy and Rellich type inequalities with detailed proofs.

In this paper we will consider plane domains $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$. Let $\operatorname{dist}(z, \partial\Omega)$ be the distance from the point $z = x + iy \in \Omega$ to the boundary of the domain. Let $C_0^\infty(\Omega)$ be the family of smooth complex-valued functions with compact supports in the domain $\Omega \neq \mathbb{C}$.

We will study the following variational Rellich type inequality: for any function $f \in C_0^\infty(\Omega)$

$$\iint\limits_{\Omega} \frac{|\Delta f(z)|^2}{(\operatorname{dist}(z, \partial\Omega))^{-2+2\mu}} dx\, dy \geq c_{2\mu}(\Omega) \iint\limits_{\Omega} \frac{|f(z)|^2}{(\operatorname{dist}(z, \partial\Omega))^{2+2\mu}} dx\, dy, \tag{1}$$

where $z = x + iy \in \Omega$, Δ denotes the Laplace operator, μ is a fixed real number, the constant $c_{2\mu}(\Omega) \in [0, \infty)$ is defined to be maximal, i.e. it is defined by the following formula

$$c_{2\mu}(\Omega) = \inf_{f \in C_0^\infty(\Omega), f \not\equiv 0} \frac{\iint_\Omega |\Delta f(z)|^2 (\operatorname{dist}(z, \partial\Omega))^{2-2\mu} dx\, dy}{\iint_\Omega |f(z)|^2 (\operatorname{dist}(z, \partial\Omega))^{-2-2\mu} dx\, dy}.$$

Notice that $c_{2\mu}(\Omega)$ is invariant under linear conformal transformation. More precisely, one has that

$$c_{2\mu}(\Omega) = c_{2\mu}(a\Omega + b), \tag{2}$$

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